MATH 5C - TEST 4 v1

Chapter 16

Fall 2024

100 POINTS NAME: ____________________

Show all work neatly. Name any theorems used.

(1) Match the equation to the vector field plot. Vector fields have been uniformly scaled in order to be seen more clearly. Two plots do not have a match. (8pts)

(8pts)

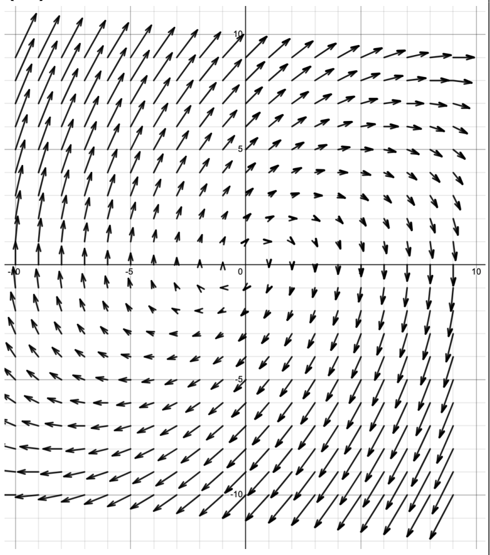
a) $\vec{F}(x,y)=\langle -x,-y\rangle$ ___6___

b) $\vec{F}(x,y)=\langle 5\sin x, x\rangle$ ___2___

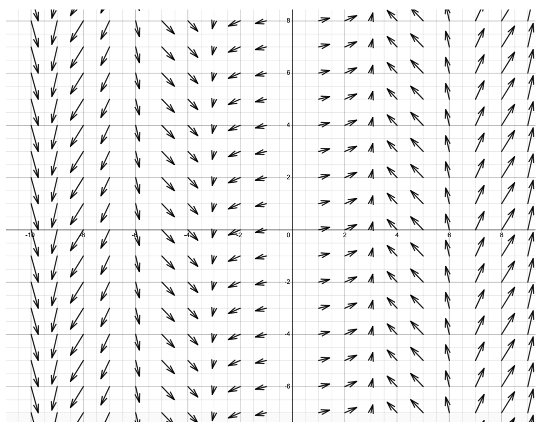
c) $\vec{F}(x,y)=\langle -x, y\rangle$ ___4___

d) $\vec{F}(x,y)=\langle y^3, x\rangle$ ___5___

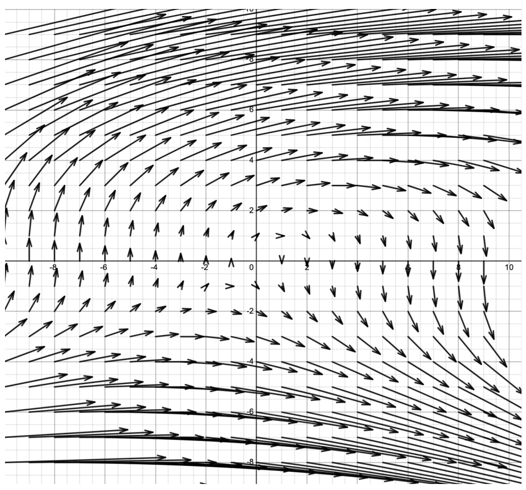
(1)



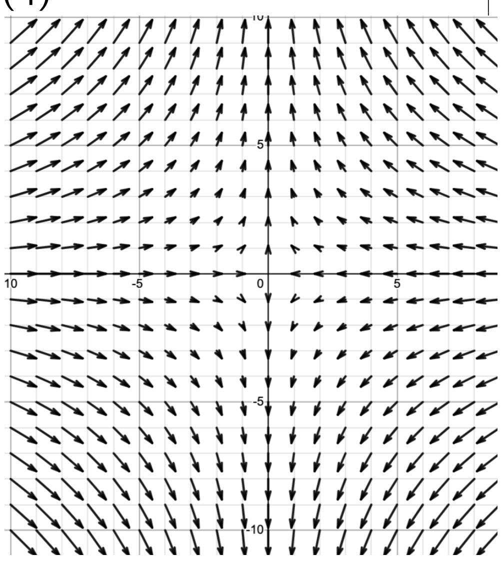
(2)



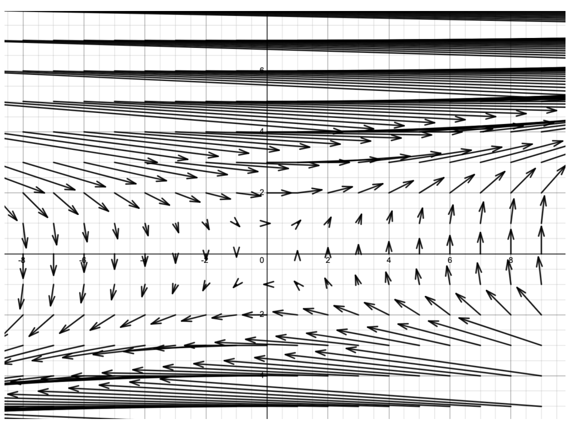
(3)



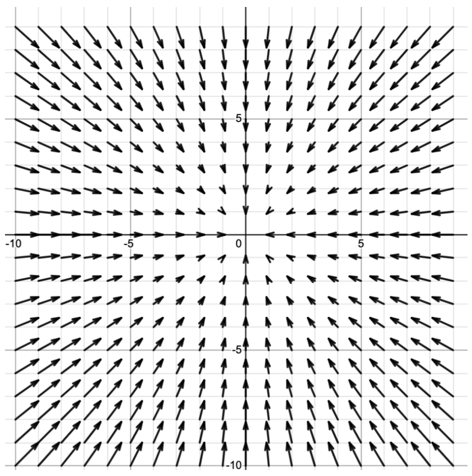
(4)



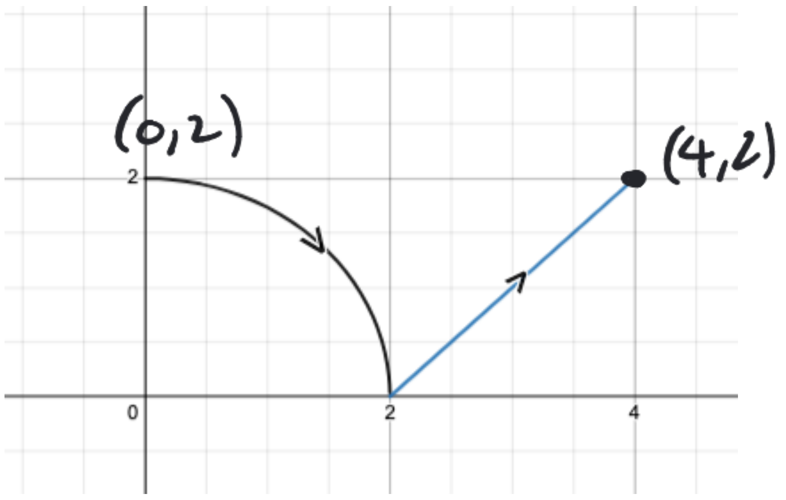
(5)



(6)



(2) Given $\vec{F}(x,y) = \langle 2x + y, x + 3y^2 \rangle$ and the piecewise smooth path C given by a quarter circle of radius 2, traveled from (0,2) to (2,0), followed a line segment from (2,0) to (4,2) (26 points)



a) Find the potential function $f(x,y)$ such that $\vec{\nabla} f(x,y) = \vec{F}(x,y)$ and use it to compute $\int_C \vec{F} \cdot d\vec{r}$

$$f(x,y) = x^2 + xy + y^3 + c$$

$$\int_C \vec{F} \cdot d\vec{r} = f(4,2) - f(0,2) = 32 - 8 = 24$$

b) Find $\int_C \vec{F} \cdot d\vec{r}$ using a **different method.** Explain.

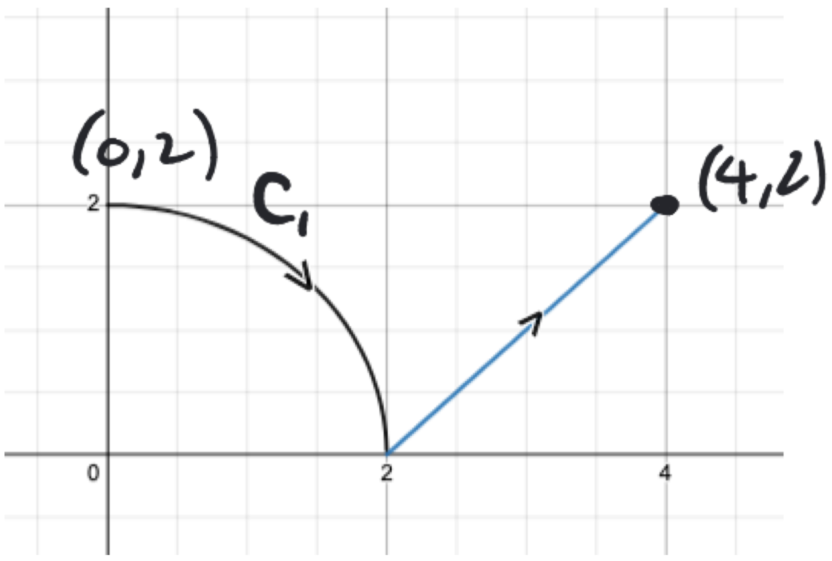
$\vec{F}$ conserv. $\Rightarrow$ use simpler path.

Line segment
$(0,2) \to (4,2)$
$x = 4t$
$y = 2$
$0 \le t \le 1$

$$\vec{F} = \langle 2(4t) + 2, 4t + 3(2)^2 \rangle$$
$$\vec{F} = \langle 8t + 2, 4t + 12 \rangle$$
$$\vec{r}\,' = \langle 4, 0 \rangle$$
$$\vec{F} \cdot \vec{r}\,' = 32t + 8$$
$$\int_0^1 (32t + 8)\,dt = 16t^2 + 8t \Big]_0^1 = 24$$

see next page for direct

Given $\vec{F}(x,y) = \langle 2x+y, x+3y^2 \rangle$ and the piecewise smooth path C given by a quarter circle of radius 2, traveled from (0,2) to (2,0), followed a line segment from (2,0) to (4,2) (26 points)



C_1:

$x = 2\sin t$

$y = 2\cos t$

$0 \le t \le \pi/2$

$\vec{F} = \langle 4\sin t + 2\cos t, 2\sin t + 12\cos^2 t \rangle$

$\vec{r}' = \langle 2\cos t, -2\sin t \rangle$

$\vec{F}\cdot\vec{r}' = 8\sin t\cos t + 4\cos^2 t - 4\sin^2 t - 24\sin t\cos^2 t$

$\int_0^{\pi/2} (8\sin t\cos t + 4\cos^2 t - 4\sin^2 t - 24\sin t\cos^2 t)\,dt$

$4(\cos^2 t - \sin^2 t)$

$4\cos 2t$

$\int_0^{\pi/2} (8\sin t\cos t + 4\cos 2t - 24\sin t\cos^2 t)\,dt$

$u = \sin t$ $\quad u = \cos t$

$4\sin^2 t + 2\sin 2t + 8\cos^3 t \,\Big]_0^{\pi/2}$

$4 - 8 = -4$

C_2 $(2,0) \to (4,2)$

$x = 2 + 2t$

$y = 2t$

$0 \le t \le 1$

$\vec{F} = \langle 4 + 6t, 2 + 2t + 12t^2 \rangle$

$\vec{r}' = \langle 2, 2 \rangle$

$\vec{F}\cdot\vec{r}' = 12 + 16t + 24t^2$

$\int_0^1 12 + 16t + 24t^2\,dt$

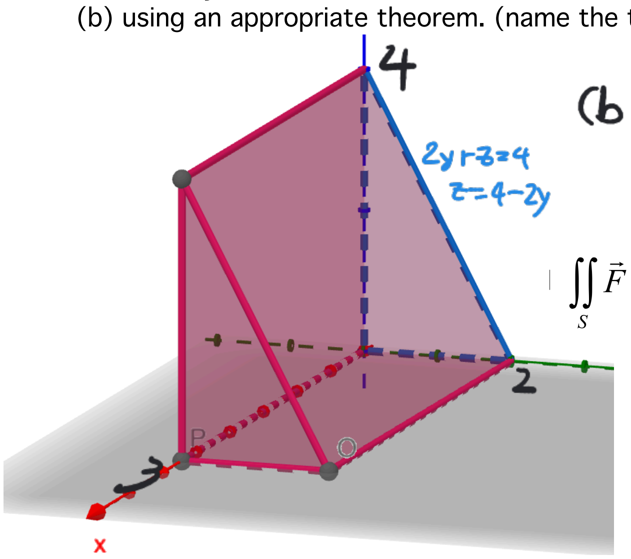
$12 + 8 + 8$

28

$\int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} = 28 - 4 = 24$

(3) Evaluate the flux integral $\iint_S \vec{F} \bullet d\vec{S} = \iint_S \vec{F} \bullet \vec{n}\, dS$ where $\vec{F}(x,y,z) = \langle y, 0, z \rangle$ and S is the closed surface formed by the plane $2y + z = 4$ the coordinate planes and the plane x=3. (outward unit normals), two ways:

(a) directly, and (16 points)
(b) using an appropriate theorem. (name the theorem) (13 points)



(b) Divergence Theorem

$\operatorname{div} \vec{F} = 1$

$$\iint_S \vec{F} \bullet d\vec{S} = \iint_S \vec{F} \bullet \vec{n}\, dS = \iiint_E dV$$

$$= \text{Volume} = \frac{1}{2}\cdot 4 \cdot 2 \cdot 3 = \boxed{12}$$

a)

Front

$x = 3$

$\vec{\nabla} G = \langle 1, 0, 0 \rangle$

$\vec{F} = \langle y, 0, z \rangle$

$\vec{F} \cdot \vec{\nabla} G = y$

$\iint_D y\, dA + \underbrace{\iint_D -y\, dA}$

$\int_0^2 \int_0^{4-2y} y\, dz\, dy$

$\int_0^2 (4y - 2y^2)\, dy$

$2y^2 - \frac{2}{3}y^3 \Big]_0^2$

$8 - \frac{16}{3}$

Back

$x = 0$

$-\vec{\nabla} G = \langle -1, 0, 0 \rangle$

$\vec{F} = \langle y, 0, z \rangle$

Left

$y = 0$

$-\vec{\nabla} G = \langle 0, -1, 0 \rangle$

$\vec{F} = \langle 0, 0, z \rangle$

$\vec{F} \cdot -\vec{\nabla} G = 0$

Top

$z = 4 - 2y$

$2y + z = 4$

$G(x,y,z)$

$\vec{\nabla} G = \langle 0, 2, 1 \rangle$

$\vec{F} = \langle y, 0, 4-2y \rangle$

$\vec{F} \cdot \vec{\nabla} G = 4 - 2y$

$\int_0^3 \int_0^2 (4 - 2y)\, dy\, dx$

$\int_0^3 4y - y^2 \Big]_0^2\, dx$

$4 \cdot 3$

12

Bottom

$z = 0$

$-\vec{\nabla} G = \langle 0, 0, -1 \rangle$

$\vec{F} = \langle y, 0, 0 \rangle$

$\vec{F} \cdot \vec{\nabla} G = 0$

$$\frac{8}{3} + \frac{-8}{3} + 0 + 12 + 0$$

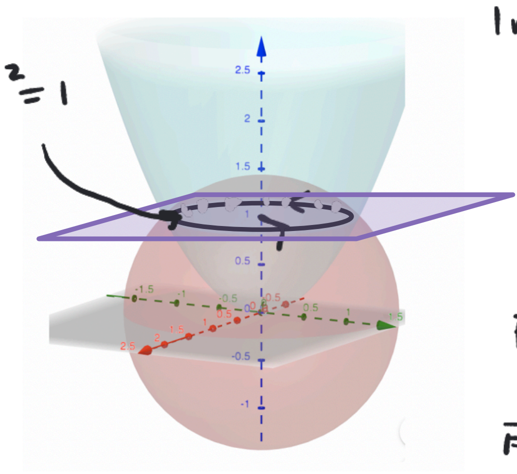
$$\boxed{12}$$

(4) Given $\vec{F}(x,y,z) = (x-y)\vec{i} + (y-z)\vec{j} + (z-x)\vec{k}$ and C is the curve of intersection of the sphere $x^2+y^2+z^2=2$ and the paraboloid $z=x^2+y^2$, oriented in the positive direction, find $\int_C \vec{F} \bullet d\vec{r}$ two ways:

(a) directly, and
(b) using another method.

(24 points)

$x^2+y^2=1$



$z=1$

Intersection: $\begin{cases} z=x^2+y^2 \\ x^2+y^2+z^2=2 \end{cases} \Rightarrow$

$z+z^2=2$

$z^2+z-2=0$

$(z+2)(z-1)=0$

$z=1 \Rightarrow x^2+y^2=1$

Parameterize C:

$\vec{r} = \langle \cos t, \sin t, 1 \rangle \quad 0 \le t \le 2\pi$

$\vec{F}(\vec{r}(t)) = \langle \cos t - \sin t, \sin t - 1, 1-\cos t \rangle$

$\vec{r}\,'(t) = \langle -\sin t, \cos t, 0 \rangle$

$\vec{F} \cdot \vec{r}\,' = -\cos t \sin t + \sin^2 t + \sin t \cos t - \cos t$

$\vec{F} \cdot \vec{r}\,' = \sin^2 t - \cos t$

$$\int_C \vec{F} \bullet d\vec{r} = \int_0^{2\pi} (\sin^2 t - \cos t)\,dt = \frac{1}{2}t - \frac{1}{4}\sin 2t - \sin t \Big]_0^{2\pi} = \boxed{\pi}$$

Use stokes thm with either of the given surfaces or with $z=1$, easiest.

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x-y & y-z & z-x \end{vmatrix} = \langle 1, 1, 1 \rangle$$

surface $z=1$

$\underbrace{z-1}_{G}=0$

$\vec{\nabla}G = \langle 0, 0, 1 \rangle$

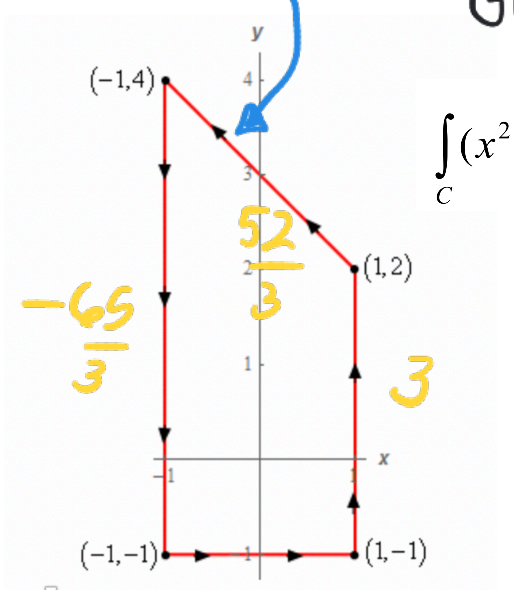
$\text{curl}\,\vec{F} = \langle 1, 1, 1 \rangle$

$$\int_C \vec{F} \bullet d\vec{r} = \iint_S \text{curl}\,\vec{F} \cdot d\vec{S} = \iint_D \text{curl}\,\vec{F} \cdot \vec{\nabla}G\,dA = \iint_D 1\,dA = \boxed{\pi}$$

(5) Find $\int_C (x^2 - xy)\,dx + y^2\,dy$ where C is the path shown below, oriented in the positive direction (Use any appropriate method)

(13 points)

$y = {}^-x + 3$



$\frac{52}{3}$

$-\frac{65}{3}$

3

$\frac{2}{3}$

If directly

Greens :

$$\int_C (x^2 - xy)\,dx + y^2\,dy = \iint_D x\,dA$$

$$= \int_{-1}^{1}\int_{-1}^{-x+3} x\,dy\,dx$$

$$= \int_{-1}^{1} x\left(-x+3-{}^-1\right)dx$$

$$= \int_{-1}^{1} -x^2 + \overset{0 \text{ odd}}{\cancel{4x}}\,dx$$

$$= {}^-\frac{1}{3}x^3\Big]_{-1}^{1} = {}^-\frac{2}{3}$$